

Volume forms for time orientable Finsler spacetimes*

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Abstract

The paper proposes extensions of the notions of Busemann-Hausdorff and Holmes-Thompson volume to time orientable Finslerian spacetime manifolds.

These notions are designed to also make sense in cases when the Finslerian metric tensors are either not defined or degenerate along some directions in each tangent space - which is the case with the majority of Lorentzian Finsler metrics used in applications. This feature makes it possible to build well-defined field-theoretical integrals having such metrics as a background.

Keywords: Lorentz-Finsler manifold, time orientability, volume form

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1 Introduction

Finsler spaces represent a natural geometric framework for applications in physics and biology. Among these, field-theoretical applications have a peculiar importance and are the most numerous (just a few examples: [1], [3], [6], [11], [12], [13], [14], [15], [18], [20], [21]). But, these applications generally require metrics to be of Lorentzian signature. And, while positive definite Finsler metrics are quite well studied and understood, there are basic geometric questions to still be clarified on Finslerian spacetime metrics, on the answer of which depend most (if not all) field-theoretical applications.

One of these basic questions (which has so far remained, to our knowledge, an open one) is the construction of a well-defined volume form for Lorentz-Finsler manifolds, to be determined from the Finsler metric alone - and which should reduce to the Riemannian volume form in the particular case of Riemannian spacetime metrics. The extension of the classical notions of volume for

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positive definite Finsler metrics, i.e., the Busemann-Hausdorff and the Holmes-Thompson ones, [16], to Lorentzian signature is far from trivial, for at least two reasons:

- The first problem is that the definitions of these volume forms both involve the integration of some function on the Finslerian unit balls (or, equivalently, on the indicatrices) of the given metric. In the case of positive definite Finsler metrics, these unit balls are compact, leading to finite integrals and therefore, to well-defined notions, but, in Lorentzian signature, they become non-compact (just an example: even in the case of the Minkowski metric $\eta = \text{diag}(1, -1, -1, -1)$ on \mathbb{R}^4 , the closed "unit ball" $\{(y^i) \in \mathbb{R}^4 \mid \eta_{ij}y^iy^j \leq 1\}$ is, actually, the interior of a hyperboloid), thus leading to infinite values of the respective integrals.

- Another serious issue is that, for most of the Lorentzian Finsler functions used in applications, such as Randers, m -th root or Bogoslovsky ones, there exist entire directions in each tangent space along which the metric tensor cannot be defined or is degenerate (this issue is even mentioned in [17] as an impediment to building physical theories based on Finsler metrics).

The paper proposes extensions of the notions of Busemann-Hausdorff and Holmes-Thompson volume to time orientable Finslerian spacetime manifolds, meant to also make sense in cases when the corresponding metric tensor is rather ill-behaved.

The technique is the following. We look for a positive definite Riemannian metric tensor to be canonically attached to a given Lorentz-Finsler metric tensor g on a manifold M ; if such a metric can be found, each of its closed unit balls E_x , $x \in M$, is an ellipsoid (hence, compact) and can be used, instead of the Finslerian unit balls, in Busemann-Hausdorff and Holmes-Thompson-type constructions.

This positive definite Riemannian metric is determined as follows:

Step 1. Each time orientation on M (regarded, as in [19], [14], as a section $x \mapsto t_x$ of the tangent bundle¹ (TM, π, M)) gives rise to a Riemannian spacetime metric g^t on M , by the rule $g_x^t := g(x, t_x)$, $\forall x \in M$. Further, using the time orientation t , we can attach to g^t a positive definite Riemannian metric $g^{t,+}$; this is possible using a known trick in general relativity, [7].

Step 2. Since the time orientation t is generally not unique, we can find multiple positive definite metrics $g^{t,+}$ and, accordingly, to non-unique outcomes for the obtained Busemann-Hausdorff and Holmes-Thompson type expressions - which is, of course, unacceptable. In order to solve this ambiguity, we will try to pick a *privileged* time orientation t_0 , which provides a minimal critical value t_0 for the functional

$$t \mapsto S_D(t) = \int_D \sqrt{\det g^{t,+}(x)} d^n x,$$

¹In the literature, there exist two nonequivalent notions of Finslerian time orientation, referring either to sections of the pullback bundle π^*TM , [3], or to sections of TM , [19], [14]. Here, it was advantageous to use the latter.

providing the $g^{t,+}$ -Riemannian volume of an arbitrary compact domain $D \subset M$.

Under the assumption that at least a privileged time orientation t_0 exists on M , then the Riemannian volume form $\sqrt{\det g^{t_0,+}(x)} d^n x$ does not depend on the choice of the privileged time orientation t_0 and represents a well-defined volume form on M , which we call the *minimal Riemannian* volume form. This volume form can actually be obtained by means of a Busemann-Hausdorff type procedure, in which the role of the Finslerian unit balls is taken by the Riemannian unit balls $E_x^{t_0}$.

If, moreover, the determinant $\det g$ is smooth and nonzero on the entire slit tangent bundle, then a Holmes-Thompson-type volume form is also uniquely defined using a privileged time orientation.

Just as in the positive definite case, the Holmes-Thompson volume is tightly connected to a certain volume form on the tangent bundle. Thus, it allows one to naturally define field-theoretical integrals also in the case when the fields under discussion depend on the fiber coordinates on TM ; an alternative construction using the minimal Riemannian volume is also briefly presented.

In the particular case of Riemannian metrics, all time orientations are privileged ones and both the above volume forms reduce to the usual Riemannian one.

The paper is structured as follows. In Section 2, we present some preliminary notions and results. In Section 3.1, we discuss the notion of time orientation on the base manifold M and, for each time orientation t , we construct a positive definite Riemannian metric $g^{t,+}$ from the initial Lorentz-Finsler metric. In Section 3.2, we introduce the notion of privileged time orientation. Section 4 is devoted to the introduction of the minimal Riemannian and of the Holmes-Thompson type volume forms. Finally, in the last section, we present three examples: smooth metrics obtained as linearized Finslerian perturbations of the Minkowski metric $\text{diag}(1, -1, -1, -1)$ on \mathbb{R}^4 , a non-smooth metric (Berwald-Moor metric) for which it is still possible to define both volume forms and, finally, a Bogoslovsky-type metric, for which we can only determine the minimal Riemannian volume form.

2 Preliminaries

1. Pseudo-Finsler and Finsler spacetimes

At present, there exist several different definitions of Finslerian spacetimes; a recent review thereof is given, e.g., in [14]. A part of them are based on a definition by Asanov, [1], and rely on a 1-homogeneous Finslerian fundamental function (norm) F , while the others are relaxed versions of a definition by Beem, [4], based on a 2-homogeneous function L . In the following, we will prefer the latter, which give a Finslerian generalization $ds^2 = L(x, dx)$ of the notion of relativistic interval and allow one to naturally introduce the notions of lightlike or spacelike vectors on the base manifold M .

The definition of pseudo-Finsler spaces we will present below is the one in [5] and includes all the usual examples of Finslerian spacetime metrics.

Let M be a connected, orientable, \mathcal{C}^∞ -smooth manifold of dimension n and (TM, π, M) , its tangent bundle. The set of sections of any fibered manifold E over M will be denoted by $\Gamma(E)$ and the set of \mathcal{C}^∞ -smooth functions on E , by $\mathcal{F}(E)$. By TM° , we will mean the slit tangent bundle $TM \setminus \{0\}$.

We denote by $(x^i)_{i=\overline{0, n-1}}$ the coordinates of a point $x \in M$ in a local chart (U, φ) . Each choice of a basis $\{b_i\}$ on $T_x M$ gives rise to the coordinate n -uple (y^i) for any vector $y \in T_x M$ (the basis $\{b_i\}$ can be the natural one $\{\partial/\partial x^i\}$, but this is not necessary, [2], p. 3.). This way, we obtain, for a point $(x, y) \in \pi^{-1}(U) \subset TM$, the coordinates $(x^i, y^i)_{i=\overline{0, n-1}}$. Whenever possible, we will make no distinction between $(x, y) \in TM$ and its coordinates $(x^i, y^i) \in \mathbb{R}^{2n}$.

In the following, we will only admit positively oriented bases $\{b_i\}$ and orientation-preserving coordinate changes $(x^i) \mapsto (x^{i'})$ on M , i.e., coordinate changes with $\det(\frac{\partial x^{i'}}{\partial x^j}) > 0$. We denote by $\{\theta^i\}$ the elements of the dual basis to $\{b_i\}$.

Consider a non-empty open submanifold $A \subset TM$, with $\pi(A) = M$ and $0 \notin A$. We assume that each $A_x := T_x M \cap A$, $x \in M$, is a positive conic set, i.e., $\forall \alpha > 0, \forall y \in A_x : \alpha y \in A_x$. The set A has the structure of a fibered manifold over M ; elements $y \in A_x$ are called *admissible vectors*.

Fix a natural number $0 \leq q < n$. A smooth function $L : A \rightarrow \mathbb{R}$ is said to define a *pseudo-Finsler structure* on M , if, in any induced local chart $(\pi^{-1}(U), \varphi^*)$ on TM and at any point $(x, y) \in A \cap \pi^{-1}(U)$:

- 1) $L(x, \alpha y) = \alpha^2 L(x, y)$, $\forall \alpha > 0$;
- 2) $g_{ij} := \frac{1}{2} \frac{\partial^2 L}{\partial y^i \partial y^j}$ are the components of a quadratic form with q negative eigenvalues and $n - q$ positive eigenvalues.

The *Finslerian energy* L can always be prolonged by continuity as 0 at $y = 0$.

In particular, if $q = 0$, then the Finsler structure (M, L) is called *positive definite*. If $q = n - 1$, then (M, L) is called a *Lorentz-Finsler space* or a *Finsler spacetime*. If $A = TM^\circ$, then (M, L) is called *smooth*. Usually, by a *Finsler structure*, one automatically understands a smooth, positive definite one, e.g., [2] - but, here, we will specify this explicitly each time. (M, L) is *(pseudo)-Riemannian*, if, in any local chart, $g_{ij} = g_{ij}(x)$ and *locally Minkowskian* if around any point of A , there exists a local chart in which $g_{ij} = g_{ij}(y)$ only.

The arc length of a curve $c : t \in [a, b] \mapsto (x^i(t))$ on M is calculated as $l(c) = \int_a^b F(x(t), \dot{x}(t)) dt$, where the Finslerian norm $F : A \rightarrow \mathbb{R}$ is given by:

$$F = \sqrt{|L|}. \quad (1)$$

The Finslerian metric tensor g can be regarded as a mapping $g : A \rightarrow T^*M \otimes T^*M$. More precisely, let us fix a local chart (U, φ) on M and $x \in U$; for each $y = y^k b_k \in A_x$, we have a symmetric bilinear form $g_{(x,y)}$ on $T_x M \simeq \mathbb{R}^n$, given, in the basis $\{b_i\}$, by the matrix $g(x, y) := (g_{ij}(x, y))$ i.e.,

$$g_{(x,y)}(b_i, b_j) := g_{ij}(x, y). \quad (2)$$

Another important quantity in a Finsler space is the *Cartan form* $\mathbf{C} = C_i(x, y)\theta^i$, with coefficients $C_i = \frac{1}{2}g^{jk}\frac{\partial g_{ij}}{\partial y^k} \in \mathcal{F}(TM)$. The coefficients C_i are related to $\det(g)$ by:

$$\frac{\partial \sqrt{\det(g)}}{\partial y^i} = C_i \sqrt{\det(g)}. \quad (3)$$

If (M, L) is Riemannian, then \mathbf{C} identically vanishes.

Remark. For smooth, positive definite Finsler metrics, the converse also holds true, i.e., if \mathbf{C} identically vanishes, then (M, L) is Riemannian (Deicke's Theorem, [2]). Still, for Lorentz-Finsler metrics, Deicke's Theorem is no longer valid. A counterexample is presented below, in Section 5.2.

2. Volume forms for smooth, positive definite Finsler metrics

A volume form ω on a manifold M is a nowhere zero n -form on M :

$$\omega = \sigma(x) d^n x, \quad (4)$$

where $d^n x := dx^0 \wedge \dots \wedge dx^{n-1}$. With respect to orientation-preserving coordinate changes $(x^i) \mapsto (x^{i'})$, the functions $\sigma(x)$ transform as:

$$\sigma(x) = \det\left(\frac{\partial x^{i'}}{\partial x^j}\right) \sigma'(x'). \quad (5)$$

More generally, a volume form can be expressed as a nonzero multiple of the exterior product $\theta^0 \wedge \theta^1 \wedge \dots \wedge \theta^{n-1}$, where $\theta^i = \theta^i(x)$ are the elements of an arbitrary basis of $\Gamma(T^*M)$. Once a volume form is defined, integrals of functions on compact domains $D \subset M$ are defined via partitions of unity.

In particular:

- The *Euclidean volume form* on \mathbb{R}^n is $d^n x$. The Euclidean volume of a compact domain $D \subset M$ is denoted by $Vol(D)$.
- On pseudo-Riemannian manifolds (M, g) , the *Riemannian volume form* is expressed in an arbitrary basis $\{\theta^i\}$ as, [9],

$$dV_g = \sqrt{|\det g(x)|} \theta^0 \wedge \theta^1 \wedge \dots \wedge \theta^{n-1}, \quad (6)$$

where $g(x)$ is the matrix of g in the dual basis $\{b_i\}$ of $\{\theta^i\}$.

Now, assume that $A = TM^\circ$ and the Finsler structure (M, L) is positive definite. Fix a local chart (U, φ) of M and an arbitrary point $x \in U$. Let $\{b_i\}$

be a positively oriented basis of $T_x M$, with dual $\{\theta^i\}$ and $y = y^i b_i \in T_x M \mapsto (y^i) \in \mathbb{R}^n$, the corresponding coordinate isomorphism.

The *closed Finslerian unit ball*,

$$B_x = \{(y^i) \in \mathbb{R}^n \mid F(x, y) \leq 1\} \quad (7)$$

is a compact, convex subset of \mathbb{R}^n .

Integrals of homogeneous functions on a Finslerian unit ball B_x and its boundary (the *indicatrix*) ∂B_x are related, [8], as follows. If $f : TM^o \rightarrow \mathbb{R}$, $(x, y) \mapsto f(x, y)$ is of class \mathcal{C}^∞ and homogeneous of degree k in y , then

$$\int_{B_x} f(x, y) d^n y := \lim_{\varepsilon \rightarrow 0} \int_{\varepsilon \leq F(x, y) \leq 1} f(x, y) d^n y$$

is well defined and

$$\int_{\partial B_x} f \lambda = (n + k) \int_{B_x} f(x, y) d^n y, \quad (8)$$

where λ denotes the Euclidean volume form on ∂B_x .

The **Busemann-Hausdorff volume form** of (M, L) is given, in the basis $\{\theta^i\}$ of $\Gamma(T^*M)$, [16], as:

$$dV_{BH} = \sigma_{BH}(x) \theta^0 \wedge \dots \wedge \theta^{n-1}, \quad \sigma_{BH}(x) = \frac{\text{Vol}(\mathbb{B})}{\text{Vol}(B_x)}, \quad (9)$$

where \mathbb{B} denotes the Euclidean unit ball in \mathbb{R}^n .

In the case when L is reversible, i.e., $L(x, y) = L(x, -y)$, $\forall (x, y) \in TM^o$, dV_{BH} gives the Hausdorff measure of the distance function induced by $F = \sqrt{L}$.

The **Holmes-Thompson volume form** of (M, L) is, [16]:

$$dV_{HT} = \sigma_{HT}(x) \theta^0 \wedge \dots \wedge \theta^{n-1}, \quad \sigma_{HT}(x) = \frac{1}{\text{Vol}(\mathbb{B})} \int_{B_x} \det g(x, y) d^n y, \quad (10)$$

with $d^n y := dy^0 \wedge dy^1 \wedge \dots \wedge dy^{n-1}$.

Particular case: If (M, g) is Riemannian, then: $dV_{BH} = dV_{HT} = dV_g$.

Remark. The fact that, for smooth, positive definite Finsler metrics, the unit balls B_x , $x \in M$, are compact, is essential for both the above definitions. Also, (10) uses the fact that $\det(g)$ is defined (and smooth) on the entire TM^o .

3 Time orientable Finsler spacetimes

3.1 Time orientations and osculating Riemannian metrics

Assume, in the following, that (M, L) is a Lorentz-Finsler manifold. An admissible tangent vector $y \in A_x$ at some point $x \in M$ is called, [19], [14]: a) *timelike*, if $L(x, y) > 0$; b) *lightlike*, if $L(x, y) = 0$ and c) *spacelike*, if $L(x, y) < 0$.

If $y \in A_x$ is timelike, then $L(x, y) = F^2(x, y)$.

Definition 1 [19]: *A time orientation on M is a smooth vector field $t \in \Gamma(A)$, which is everywhere timelike.*

If the Finsler spacetime (M, L) admits a time orientation, then it is called *time orientable*.

In the following, we will always assume that (M, L) is time orientable².

Consider the subset of A consisting of timelike vectors:

$$A^+ := A \cap L^{-1}(0, \infty). \quad (11)$$

The open subset $A^+ \subset A$ is a submanifold of A . Moreover, since (M, L) is time orientable, there exists at every $x \in M$, at least a timelike vector, i.e., $\pi(A^+) = M$. Consequently, A^+ has the structure of a fibered manifold over M . Time orientations t can thus be regarded as (smooth) sections of A^+ .

Let $t \in \Gamma(A^+)$, $x \mapsto t_x$ denote an arbitrary time orientation on M . Then, the mapping $x \in M \mapsto g_x^t \in T^*M \otimes T^*M$, given by:

$$g_x^t := g_{(x, t_x)}, \quad \forall x \in M, \quad (12)$$

defines a pseudo-Riemannian metric g^t on M , called, [18], an *osculating Riemannian metric*³ of the Lorentz-Finsler metric g .

Remark. The fact that t is everywhere admissible ensures that $g_x^t(v, w)$ is well defined for any vectors $v, w \in T_x M$ and the dependence $x \mapsto g_x^t$ is smooth (even if $A \subsetneq TM^o$, i.e., if the initial Finsler metric $g = g(x, y)$ is ill-behaved along certain directions $y \in T_x M$). That is, g^t is, indeed, a well-defined pseudo-Riemannian metric.

Now, fix a time orientation $t \in \Gamma(A^+)$. The osculating Riemannian metric g^t , $t \in \Gamma(A)$ has Lorentzian signature $(+, -, -, \dots, -)$. Following the model in [7]

²The vast majority of "reasonable" physical models are based on time orientable spacetimes. The case of non-time-orientable spacetimes can be dealt with, e.g., by passing to a double covering space, [10].

³The term *osculating Riemannian metric* is also used in the Finsler literature (e.g., [8]) with a different meaning, i.e., a Riemannian metric obtained by an averaging technique. Here, we do *not* have in mind this meaning (averaging techniques are not even available until a volume form is defined).

(Remark 2.4, Ch. XII), we define the mapping $g^{t,+} : M \rightarrow T^*M \otimes T^*M$, $x \mapsto g_x^{t,+}$, with⁴:

$$g_x^{t,+}(v, w) := 2g_x^t(t'_x, v)g_x^t(t'_x, w) - g_x^t(v, w), \quad \forall v, w \in T_x M, \quad (13)$$

where $t' := \frac{t}{F(t)}$. In local writing, we have, at any $x \in M$:

$$g_{ij}^{t,+} = 2t'_i t'_j - g_{ij}^t, \quad (14)$$

where $t'_i = g_{ij}^t t'^j$.

Proposition 2 *i) Given any time orientation $t \in \Gamma(A^+)$, the metric $g^{t,+}$ is a positive definite Riemannian metric on M .*

ii) Corresponding to any local chart, there holds the equality:

$$\det(g^{t,+}) = |\det(g^t)|. \quad (15)$$

Proof. *i)* Each $g_x^{t,+}$, $x \in M$, is a symmetric bilinear form on $T_x M$ and the dependence $x \mapsto g_x^{t,+}$ is smooth, i.e., $g^{t,+}$ is a pseudo-Riemannian metric.

Let us also check that each $g_x^{t,+}$ is positive definite. Fix an arbitrary $x \in M$ and pick a g_x^t -orthonormal basis $\{\hat{e}_i\}_{i=\overline{0, n-1}}$ on $T_x M$, with $\hat{e}_0 = t'$; that is, $g_x^t(\hat{e}_i, \hat{e}_j) = \eta_{ij}$, where $\eta = \text{diag}(1, -1, -1, \dots, -1)$. For any $v = v^i \hat{e}_i \in T_x M$, we have: $g_x^t(t', v) = g_x^t(\hat{e}_0, v) = v^0$, $g^t(v, v) = (v^0)^2 - (v^1)^2 - \dots - (v^{n-1})^2$ and:

$$g_x^{t,+}(v, v) = 2[g_x^t(t'_x, v)]^2 - g_x^t(v, v) = (v^0)^2 + (v^1)^2 + \dots + (v^{n-1})^2 \geq 0.$$

Moreover, $g_x^{t,+}(v, v) = 0$ if and only if $v^i = 0$, $i = \overline{0, n-1}$, i.e., $v = 0$.

ii) We will use the following result, [2], p. 287: If (Q_{ij}) is a nonsingular $n \times n$ complex matrix with inverse (Q^{ij}) and $C_j \in \mathbb{C}$, $j = \overline{0, n-1}$, then:

$$\det(Q_{jk} + C_j C_k) = (1 + Q^{hl} C_h C_l) \det(Q_{jk}). \quad (16)$$

Fix an arbitrary point $x \in M$, a local chart around x and a basis $\{b_i\}$ of $T_x M$. Set: $Q_{jk} = g_{jk}^t(x)$, $C_j = i\sqrt{2}t'_j(x) \in \mathbb{C}$. Taking into account (14), we have, at x :

$$\det(g^{t,+}) = (-1)^n \det(g_{ij}^t - 2t'_i t'_j) = (-1)^n (1 - 2(g^t)^{ij} t'_i t'_j) \det(g^t)$$

With $(g^t)^{ij} t'_i t'_j = L(t') = 1$, we get $\det(g^{t,+}) = (-1)^{n+1} \det(g^t) = |\det(g^t)|$. ■

Fix $x \in M$, a basis $\{b_i\}$ of $T_x M$ and denote by (y^i) the coordinates of $y \in T_x M$ in this basis. If t is a time orientation on M , the closed unit ball of $g^{t,+}$ at x is the ellipsoid

$$E_x^t = \{(y^i) \in \mathbb{R}^n \mid g_x^{t,+}(y, y) \leq 1\}. \quad (17)$$

⁴The signs in (13) differ from the ones in [7] due to different metric signature conventions.

The ellipsoid (17) is the image of the Euclidean unit ball $\mathbb{B} = \{(u^{i'}) \in \mathbb{R}^n \mid \delta_{i'j'} u^{i'} u^{j'} \leq 1\}$ through a linear transformation $\varphi : \mathbb{R}^n \rightarrow \mathbb{R}^n$, $(u^{i'}) \mapsto (y^i)$, given by:

$$y^i = a_{j'}^i u^{j'}, \quad (18)$$

with Jacobian determinant:

$$\det(a_{j'}^i) = [\det g^{t,+}(x)]^{-1/2} > 0. \quad (19)$$

(The matrix $(a_{j'}^i)$ is determined as the matrix of change of basis from the initial basis $\{b_i\}$ to a positively oriented, $g^{t,+}$ -orthonormal basis (e'_i) with corresponding coordinates denoted by $(u^{i'}) \in \mathbb{R}^n$).

As a consequence, Euclidean volume $\text{Vol}(E_x^t) = \int_{E_x^t} d^n y$ is:

$$\text{Vol}(E_x^t) = \frac{\text{Vol}(\mathbb{B})}{\sqrt{\det g^{t,+}(x)}} = \frac{\text{Vol}(\mathbb{B})}{\sqrt{|\det g^t(x)|}}. \quad (20)$$

3.2 Privileged time orientation

The time orientation of a given Lorentz-Finsler manifold is, generally, far from unique - and different time orientations $t \in \Gamma(A^+)$ give rise to different Riemannian volume forms dV_{g^t} . In the following, we will try to pick a time orientation which, roughly speaking, *minimizes* the Riemannian volume of an arbitrary compact domain $D \subset X$.

Fix an arbitrary compact domain $D \subset M$. The functional $S_D : \Gamma(A^+) \rightarrow \mathbb{R}$, defined by:

$$S_D(t) := \int_D dV_{g^{t,+}} = \int_D \sqrt{|\det g_x^t|} d^n x, \quad (21)$$

where $g_{ij}^t(x) = g_x^t(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$, is invariant to arbitrary coordinate changes on M . Also, S_D admits an infimum (as $S_D(t) > 0, \forall t \in \Gamma(A^+)$).

A global minimum for S_D on the (open) set A^+ is not guaranteed to exist, as S_D might decrease to its infimum as we approach, e.g., a lightlike direction (see, e.g., the example in Subsection 38). This is why, in order to increase our chances of obtaining well-defined volume forms on (M, L) , we will relax the minimality request, as follows.

Definition 3 *We call a privileged time orientation on (M, L) , any time orientation $t_0 \in \Gamma(A^+)$, such that $S_D(t_0)$ is a minimum of the set of critical values of the functional S_D in (21), for any compact domain $D \subset M$.*

Particular case (Riemannian spacetimes): If $g = g(x)$ only, then the mapping $t \mapsto S_D(t)$ is, in fact, constant. In other words, *in time orientable Riemannian spacetimes, all time orientations are privileged ones.*

Let us determine the Euler-Lagrange equations for S_D . As the Lagrangian density $\mathcal{L} := \sqrt{|\det g(x, t)|}$ in (21) is of order zero in t , critical points of S_D are given by:

$$\frac{\partial}{\partial t^i} \sqrt{|\det g(x, t)|} = 0, \quad \forall x \in M. \quad (22)$$

Using (3) and the fact that $\det g(x, t) \neq 0$, the above is equivalent to

$$C_i(x, t) = 0. \quad (23)$$

This can be reformulated as:

Proposition 4 *If $t_0 \in \Gamma(A^+)$ is a critical point of the functional S_D , then it is a zero of the Cartan form:*

$$\mathbf{C}(x, t_0(x)) = 0, \quad \forall x \in M. \quad (24)$$

4 Volume forms

4.1 Minimal Riemannian volume form

Assume, in the following, that the Lorentz-Finsler manifold (M, L) admits at least a privileged time orientation t_0 .

Let us start with the following remark. If t_0 and t'_0 are two privileged time orientations, then, for any compact $D \subset M$, they provide the same (minimal critical) value for S_D . In other words, the values $S_D(t_0)$ and $S_D(t'_0)$ have to coincide for *all* compact domains $D \subset M$. As a consequence, the corresponding Lagrangian densities (which are smooth functions) have to coincide pointwise, i.e., for any $x \in M$ and in any local chart around x ,

$$\sqrt{|\det g^{t_0}(x)|} = \sqrt{|\det g^{t'_0}(x)|}. \quad (25)$$

That is,

$$dV_{g^{t_0}} = dV_{g^{t'_0}}. \quad (26)$$

It makes thus sense

Definition 5 *We call the **minimal Riemannian volume form** on (M, L) , the differential form:*

$$dV_{bh} = dV_{g^{t_0}}, \quad (27)$$

where t_0 is any privileged time orientation for (M, L) .

Relation (26) ensures that the above definition does not depend on the choice of the privileged time orientation t_0 .

Particular case. If (M, g) is Riemannian, then dV_{bh} coincides with the Riemannian volume form dV_g .

The minimal Riemannian volume form can be also obtained via a Busemann-Hausdorff type construction, as follows. Let us denote by $\{b_i\}$ a positively oriented basis of $\Gamma(TM)$ and by $\{\theta^i\}$, the dual basis of $\Gamma(T^*M)$. In this basis, the minimal Riemannian volume element is expressed as $dV_{bh} = \sqrt{|\det g^{t_0}(x)|} \theta^0 \wedge \dots \wedge \theta^{n-1}$. Using (20), we have

$$\sqrt{|\det g^t(x)|} = \frac{Vol(\mathbb{B})}{Vol(E_x^t)}, \quad (28)$$

which leads to:

$$dV_{bh} = \sigma_{bh}(x) \theta^0 \wedge \theta^1 \wedge \dots \wedge \theta^{n-1}, \quad \sigma_{bh}(x) = \frac{Vol(\mathbb{B})}{Vol(E_x^{t_0})}. \quad (29)$$

Equation (23), together with (20) tell us that, at each point x of the base manifold, and in any local chart around x , the volume $Vol(E_x^{t_0})$ is a critical value of the real-valued mapping $t_x \mapsto Vol(E_x^t)$.

4.2 Holmes-Thompson volume form

Consider a privileged time orientation $t_0 \in \Gamma(A^+)$ on M and denote by $E_x^{t_0}$ the corresponding Riemannian unit ball at $x \in M$. We assume, in the following, that the function $\det g = \det g(x, y)$ (where, by $g_{ij}(x, y)$, we mean $g_{(x,y)}(b_i, b_j)$) can be continuously prolonged to TM^o and the prolongation, also denoted by $\det g$, is smooth and nowhere zero. Under these circumstances, it makes sense:

Definition 6 *The **Holmes-Thompson volume form** on the Lorentz-Finsler space (M, L) is the differential form:*

$$dV_{ht} = \sigma_{ht}(x) \theta^0 \wedge \theta^1 \wedge \dots \wedge \theta^{n-1}, \quad \sigma_{ht}(x) = \frac{1}{Vol(\mathbb{B})} \int_{E_x^{t_0}} |\det g(x, y)| d^n y. \quad (30)$$

Proposition 7 *dV_{ht} is a well-defined volume form on M .*

Proof. 1. *Nondegeneracy:* Fix an arbitrary atlas on TM . Taking into account that the function $|\det(g)|$ is smooth on TM^o and 0-homogeneous in y , the integral (30) is well defined and it can be expressed, using (8), as an integral on the boundary $\partial E_x^{t_0}$:

$$\int_{E_x^{t_0}} |\det g(x, y)| d^n y = \frac{1}{n} \int_{\partial E_x^{t_0}} |\det g(x, y)| \lambda.$$

As $\partial E_x^{t_0}$ is compact and $|\det g|$ is continuous on $\partial E_x^{t_0}$, the minimum

$$\min_{(y^i) \in \partial E_x^{t_0}} |\det g(x, y)| =: g_{\min}(x)$$

always exists - and, under the above assumptions, it is strictly positive. Taking, again, into account the 0-homogeneity of g , we have: $g_{\min}(x) = \min_{(y^i) \in E_x^t} |\det g(x, y)|$, therefore,

$$\sigma_{ht}(x) \geq \frac{g_{\min}(x)}{\text{Vol}(\mathbb{B})} \int_{E_x^{t_0}} d^n y = g_{\min}(x) \frac{\text{Vol}(E_x^{t_0})}{\text{Vol}(\mathbb{B})} > 0, \quad (31)$$

which proves the statement.

2. *The rule of transformation* (5) with respect to coordinate changes $(x^i) \mapsto (x^{i'})$ on M :

Fix $x \in M$. A brief computation using (2) shows that dV_{ht} is invariant to changes of bases $\{\theta^i\} \rightarrow \{\theta^{i'}\}$ on each cotangent space T_x^*M . Hence, we can take with no loss of generality $\theta^i := dx^i$, i.e., in (30), $dV_{ht} = \sigma_{ht}(x) d^n x$ and $g_{ij} = g_{(x,y)}(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j})$.

Consider an arbitrary coordinate change $(x^i) \mapsto (x^{i'})$ on M . Relative to the induced coordinate change: $x^i = x^i(x^{k'})$, $y^i = \frac{\partial x^i}{\partial x^{k'}} y^{k'}$ on TM , the functions g_{ij} transform as: $g_{ij} = \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^{j'}}{\partial x^j} g_{i'j'}$; therefore, $\det g(x, y) = [\det(\frac{\partial x^{i'}}{\partial x^j})]^2 \det g(x', y')$, which, substituted into (30), gives the result.

3. *Independence on the choice of the privileged time orientation* t_0 : Let $t_0, \tilde{t}_0 \in \Gamma(A^+)$ be two privileged time orientations. Fix a local chart (U, φ) on M , an arbitrary point $x \in U$ and a basis $\{b_i\}$ of $T_x M$. Then, each of the ellipsoids $E_x^{t_0}, E_x^{\tilde{t}_0}$ is the image of \mathbb{B} through an invertible linear transformation, as in (18). More precisely, take: $\varphi, \tilde{\varphi} : \mathbb{R}^n \rightarrow \mathbb{R}^n$, given by: $u^{i'} \mapsto y^i := a_{ij}^i u^{j'}$ and $u^{i'} \mapsto \tilde{y}^i := \tilde{a}_{ij}^i u^{j'}$ respectively, such that

$$\varphi(\mathbb{B}) = E_x^{t_0}, \quad \tilde{\varphi}(\mathbb{B}) = E_x^{\tilde{t}_0}.$$

The corresponding Jacobian determinants are as in (19), i.e.,

$$\det(a_{j'}^i) = |\det g^{t_0}(x)|^{-1/2}, \quad \det(\tilde{a}_{j'}^i) = |\det g^{\tilde{t}_0}(x)|^{-1/2}.$$

Since t_0 and \tilde{t}_0 are both privileged time orientations, we have: $|\det g^{t_0}(x)|^{-1/2} = |\det g^{\tilde{t}_0}(x)|^{-1/2}$. Therefore, the linear mapping

$$\varphi \circ \tilde{\varphi}^{-1} : \mathbb{R}^n \mapsto \mathbb{R}^n, \quad (\tilde{y}^i) \mapsto (y^j) \quad (32)$$

is volume-preserving, i.e., $\det(\frac{\partial y^i}{\partial \tilde{y}^j}) = 1$, and maps diffeomorphically $E_x^{\tilde{t}_0}$ to $E_x^{t_0}$. Taking into account that, by their definition, the functions $g_{ij}(x, y)$ behave tensorially under *linear* transformations $(\tilde{y}^i) \mapsto (y^j)$ on \mathbb{R}^n (which

can be traced back to changes of bases $\{b_i\} \rightarrow \{\tilde{b}_i\}$ on $T_x M$, we have:

$$\det g(x, y) = [\det(\frac{\partial \tilde{y}^i}{\partial y^j})]^2 \det g(x, \tilde{y}) = \det g(x, \tilde{y}) \text{ and}$$

$$\frac{1}{Vol(\mathbb{B})} \int_{E_x^{t_0}} |\det g(x, y)| d^n y = \frac{1}{Vol(\mathbb{B})} \int_{E_x^{\tilde{t}_0}} |\det g(x, \tilde{y})| d^n \tilde{y},$$

i.e. $\sigma_{ht}(x)$ does not depend on the choice of the privileged time orientation. ■

Particular case. If (M, L) is Riemannian, then, using (20), we get:

$$\sigma_{ht}(x) = \frac{|\det g(x)|}{Vol(\mathbb{B})} Vol(E_x^t) = \sqrt{|\det g(x)|},$$

i.e., the Holmes-Thompson volume form (30) reduces to the Riemannian volume form dV_g .

Field-theoretical integrals with direction dependent fields. Just as in the positive definite case ([16], p. 26), the Holmes-Thompson volume form is tightly related to a volume form on TM . This allows us to naturally introduce field-theoretical actions in the case when the fields also depend on the directional variables y^i , i.e., they are represented by sections $(x, y) \mapsto q^\sigma(x, y)$ of some fibered manifold over TM .

Consider a smooth Lagrangian function

$$\mathfrak{L}(x, y) := \mathfrak{L}(x, y, q^\sigma(x, y), q_{,i}^\sigma(x, y), q_{,i}^\sigma(x, y), \dots, q_{,i_1 \dots i_r}^\sigma(x, y))$$

on TM (where $\cdot_{,i}$ and \cdot_i denote partial differentiation with x^i and y^i respectively), which is invariant under arbitrary coordinate changes on TM . The *action* attached to \mathfrak{L} and to a compact domain $D \subset M$ can be defined⁵ as:

$$S_D(q) = \frac{1}{Vol(\mathbb{B})} \int_D \int_{E_x^{t_0}} \mathfrak{L}(x, y) |\det g(x, y)| d^n y \theta^0 \wedge \dots \wedge \theta^{n-1}. \quad (33)$$

By a similar reasoning to the one in Proposition 7, we find that the value $S_D(q)$ does not depend on the choice of the privileged time orientation t_0 .

Remark 8 *If the determinant $\det(g)$ cannot be continuously prolonged by nonzero values to the entire slit tangent bundle TM° , then (33) cannot be constructed. In this case, we can still obtain a well-defined action if we replace in (33), $|\det g(x, y)|$, by $|\det(g^{t_0}(x))|$.*

⁵A somewhat similar expression of a Finslerian action is to be found in [15]; the difference is that, in the cited paper, integration of the Lagrangian with respect to the fiber coordinates y^i is carried out on the indicatrices S_x (given by $|L| = 1$) of the initial Lorentz-Finsler metric - which are non-compact, thus leading to improper integrals. Here, these indicatrices are replaced by the compact sets $E_x^{t_0}$.

5 Examples

5.1 Linearized perturbations of the Minkowski metric

Consider, on the Minkowski spacetime (\mathbb{R}^4, η) (where $\eta = \text{diag}(1, -1, -1, -1)$), an arbitrary smooth, positive definite Finsler metric tensor $\gamma_{ij} = \gamma_{ij}(x, y)$ and a small constant $\varepsilon > 0$, with $\varepsilon^2 \simeq 0$. We define, on $T\mathbb{R}^4 \setminus \{0\}$, the function:

$$L(x, y) = \eta_{ij} y^i y^j + \varepsilon \gamma_{ij}(x, y) y^i y^j. \quad (34)$$

This gives a smooth Lorentz-Finsler structure on \mathbb{R}^4 , with metric tensor $g_{ij}(x, y) = \eta_{ij} + \varepsilon \gamma_{ij}(x, y)$. Its determinant

$$|\det(g(x, y))| = 1 + \varepsilon \eta^{ij} \gamma_{ij}(x, y) \quad (35)$$

is 0-homogeneous in y and smooth on $T\mathbb{R}^4 \setminus \{0\}$, hence, it admits a nonzero global minimum on each tangent space $T_x \mathbb{R}^4$. Privileged time orientations $t_0 \in \Gamma(T\mathbb{R}^4 \setminus \{0\})$ are solutions of (22), i.e., $\eta^{ij} \gamma_{ij \cdot k}(t) = 0$. Once a privileged time orientation is chosen, we can write:

$$dV_{bh} = \sqrt{|\det g(x, t_{0,x})|} d^4 x$$

and dV_{ht} is given by (30).

5.2 Berwald-Moor metric

Consider, on $M = \mathbb{R}^4$, a sign-adjusted version of the *Berwald-Moor* quartic Finslerian metric, [6]:

$$L = \text{sgn}(y^0 y^1 y^2 y^3) \sqrt{|y^0 y^1 y^2 y^3|}.$$

The sign $\text{sgn}(y^0 y^1 y^2 y^3)$ (which does not appear in [6]) is introduced in order to allow L to also take negative values - and hence, to be able to define L -spacelike vectors.

The corresponding metric tensor

$$(g_{ij}) = \begin{cases} -\frac{1}{8} \frac{L}{(y^i)^2}, & i = j \\ \frac{1}{8} \frac{L}{y^i y^j}, & i \neq j \end{cases} \quad (36)$$

is only defined on $T\mathbb{R}^4 \setminus \{y \mid \exists i : y^i = 0\}$ - and tends to infinity as we approach any of the hyperplanes $y^i = 0$. Still, its determinant

$$\det(g_{ij}(y)) = -2^{-8}, \quad \forall (y^i) \in \mathbb{R}^4 \quad (37)$$

is a constant and hence, admits a smooth prolongation to the entire $T\mathbb{R}^4 \setminus \{0\}$. Any time orientation t is a privileged one and gives the same value $\sqrt{|\det(g^t)|} = 2^{-4}$. Substituting into (27), the minimal Riemannian volume is:

$$dV_{bh} = 2^{-4} d^4 x.$$

The Holmes-Thompson volume is given by:

$$\sigma_{ht} = \frac{1}{\text{Vol}(\mathbb{B})} \int_{E_x^t} \frac{1}{2^8} d^m y = \frac{1}{2^8} \frac{\text{Vol}(E_x^t)}{\text{Vol}(\mathbb{B})} = 2^{-4},$$

(where we have used the equalities $\frac{\text{Vol}(E_x^t)}{\text{Vol}(\mathbb{B})} = |\det(g^t)|^{-1/2} = 2^4$). That is:

$$dV_{bh} = dV_{ht} = 2^{-4} d^4 x.$$

Remark. From (3) and (37), we find out that the Cartan form \mathbf{C} of g identically vanishes - and yet, g is non-Riemannian. This points out that positive definiteness and/or smoothness of the metric are essential hypotheses for Deicke's theorem.

5.3 A Bogoslovsky type metric

Bogoslovsky metrics, expressible as: $L = (n_i y^i)^{2b} (\eta_{jk} y^j y^k)^{1-b}$, where $b \in (0, 1)$ and $n_i \in \mathbb{R}$ are covector components, are connected to very special relativity, [6]. In the following, we will study a toy model on \mathbb{R}^2 , with $b = 1/2$:

$$L = y^0 \sqrt{|(y^0)^2 - (y^1)^2|}. \quad (38)$$

The metric tensor:

$$g(y) = \frac{1}{2|(y^0)^2 - (y^1)^2|^{3/2}} \begin{pmatrix} y^0[2(y^0)^2 - 3(y^1)^2] & (y^1)^3 \\ (y^1)^3 & -(y^0)^3 \end{pmatrix}$$

is only defined and invertible outside the lightlike directions $y^1 = \pm y^0$. Its determinant:

$$\det g(y) = -\frac{2(y^0)^2 + (y^1)^2}{4|(y^0)^2 - (y^1)^2|}$$

tends to minus infinity as we approach these axes - hence, we cannot prolong it by continuity at $y^1 = \pm y^0$; therefore, we will only determine in this case the minimal Riemannian volume element dV_{bh} .

Critical directions $t = (t^0, t^1)$ for $|\det(g)|$ are $t^0 = 0$ and $t^1 = 0$. The former cannot be used as a time orientation, since it is lightlike, that is, the only viable candidate for the privileged time orientation is $t^1 = 0$ (and $t^0 > 0$). For this direction, we find $|\det(g^t)| = 1/2$. Substituting this value into the expression of dV_{bh} , we obtain the minimal Riemannian volume element as:

$$dV_{bh} = 2^{-1/2} d^2 x.$$

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